MATHEMATICS

B.Sc. Part I, Paper II

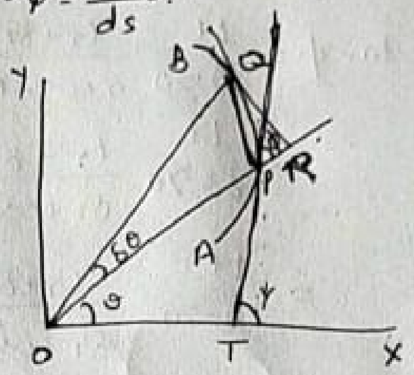
TOPIC - Tangent and Normal

J. K. Sinha

1. Angle between the radius vector and the tangent

If ϕ be the angle between the radius vector and the tangent at any point on the curve $r = f(\theta)$ then

$$\tan\phi = r\frac{d\theta}{dr};\ \sin\phi = r\frac{d\theta}{ds};\ \cos\phi = \frac{dr}{ds}.$$



Let P be any point on the curve whose polar equation is $r = f(\theta)$ and Q be another point on the curve very near to P. Let the co-ordinates of P and Q be (r, θ) and $(r+\delta r, \theta+\delta\theta)$ respectively. Let us join PQ and draw QR perpendicular to the radius vector OP produced.

Let TP be the tangent at $P(r, \theta)$ and ϕ be the angle between the tangent at P and the radius vector OP.

The following cases arise when $\delta\theta$ tends to zero in the limit.

(i) the point Q will tend to the point P.

(ii) the chord PQ will tend to the tangent TP.

(iii) the angle QPR will tend to ϕ.

Again since P and Q lie on the curve, so the equation of the curves will be

$$r = f(\theta) \text{ and } r + \delta r = f(\theta + \delta\theta).$$

Now from these two relations, we have

$$\delta r = f(\theta+\delta\theta) - f(\theta)$$

$$\frac{\delta r}{\delta\theta} = \frac{f(\theta+\delta\theta) - f(\theta)}{\delta\theta}$$

Taking limit on both sides when $\delta\theta \to 0$, we get

$$\underset{\delta\theta\to 0}{Lt}\frac{\delta r}{\delta\theta} = \underset{\delta\theta\to 0}{Lt}\frac{f(\theta+\delta\theta) - f(\theta)}{\delta\theta} = f'(\theta) = \frac{d}{d\theta}f(\theta)$$

[where dash denotes the differentiation w.r.to θ]

$$= \frac{dr}{d\theta} \qquad [\because f(\theta) = r]$$

$$\therefore \underset{\delta\theta \to 0}{Lt} \frac{\delta r}{\delta\theta} = \frac{dr}{\delta\theta}.$$

Now $\tan\phi = \underset{\delta\theta\to 0}{Lt} \tan QPR = \underset{\delta\theta\to 0}{Lt} \frac{QR}{PR}$ [from $\angle PQR$]

$$= \underset{\delta\theta\to 0}{Lt} \frac{(r+\delta r)\sin\delta\theta}{(r+\delta r)\cos\delta\theta - r} \quad \left[\because \sin\delta\theta = \frac{QR}{OQ}\right]$$

$\therefore QR = (r+\delta r)\sin\delta\theta$

and $PR = OR - OP = (r+\delta r)\cos\delta\theta - r$

$$\therefore \tan\phi = \underset{\delta\theta\to 0}{Lt} \frac{(r+\delta r)\sin\delta\theta}{r\cos\delta\theta + \delta r\cos\delta\theta - r}$$

$$= \underset{\delta\theta\to 0}{Lt} \frac{(r+\delta r)\sin\delta\theta}{\delta r\cos\delta\theta - r(1-\cos\delta\theta)}$$

$$= \underset{\delta\theta\to 0}{Lt} \frac{(r+\delta r)\sin\delta\theta}{\delta r\cos\delta\theta - r.2\sin^2\frac{\delta\theta}{2}}$$

$$= \underset{\delta\theta\to 0}{Lt} \frac{(r+\delta r)\frac{\sin\delta\theta}{\delta\theta}}{\frac{\delta r}{\delta}\cos\delta\theta - r \frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}} \cdot \sin\frac{\delta\theta}{2}}$$

$$= \frac{(r+0).1}{\frac{dr}{d\theta}.1 - r.1.0} = r\frac{d\theta}{dr}$$

$$\therefore \tan\phi = \frac{rd\theta}{dr}.$$

Also $\sin\phi = \underset{\delta\theta\to 0}{Lt} \frac{QR}{QR} = \underset{\delta\theta\to 0}{Lt} \frac{(r+\delta r)\sin\delta\theta}{QP}$

$$= \underset{\delta\theta\to 0}{Lt} \frac{r\sin\delta\theta}{\delta\theta}.\frac{\delta\theta}{\delta s}.\frac{\delta s}{QP} \quad \left[\because \delta\theta\to 0 \text{ so that } r+\delta r \to r\right]$$

$$= \underset{\delta\to 0}{Lt} r.\frac{\sin\delta\theta}{\delta\theta}.\frac{\delta\theta}{\delta s}.\frac{\text{arc } PQ}{\text{chord } PQ}.$$

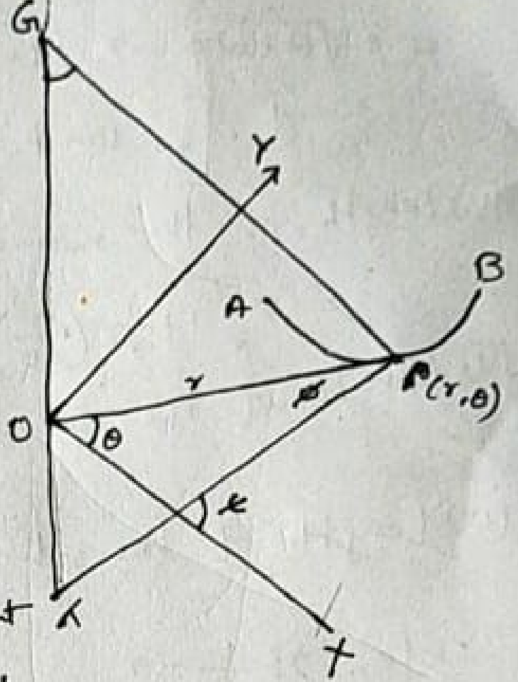
In the limit, we have

$$\sin\phi = r.1.\frac{d\theta}{ds}.1 = r\frac{d\theta}{ds}.$$

Similarly $\cos\phi = \frac{dr}{ds}$ $\left[\text{for } \cos\phi = \cot\phi.\sin\phi = \frac{dr}{rd\theta} r\frac{d\theta}{ds} = \frac{dr}{ds}\right]$

Hence $\tan\phi = \frac{rd\theta}{dr}$; $\sin\phi = \frac{rd\theta}{ds}$; $\cos\phi = \frac{dr}{ds}$.

Length of the polar tangent, normal, sub-tangent and sub-normal



Let P be any point on the curve AB whose coordinates be (r, θ).
Let PT be the tangent at P and PG be the normal at P.
Let GT be the perpendicular on the radius vector OP passing through origin O.
Let ϕ be the angle of the tangent at P which makes with the radius vector.
The portion of the tangent intercepted between the point of contact and the line passing through the pole (origin) and perpendicular to the radius vector is called the Length of the tangent (= TP) and the portion of the normal intercepted between the point of and the line passing through the pole (origin) and perpendicular to the radius vector is called the Length of the normal (= PG). The projection of the tangent of the above line is called the Length of the sub-tangent (= OT) and the projection of the normal on the above line is called the Length of the sub-normal (= OG),

From ΔOGP, we have

$$\angle GOP + \angle OPG + \angle PGO = 180^\circ$$

or, $90^\circ - \angle(90^\circ - \phi) + \angle PGO = 180^\circ$

or $90^\circ + 90^\circ - \phi + \angle PGO = 180^\circ$

$\therefore \angle PGO = \phi$.

(i) Length of the tangent $= TP = r\sec\phi = r\sqrt{\sec^2\phi}$

$$= r\sqrt{1+\tan^2\phi} = r\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}$$

$$\left[\because \tan\phi = r\frac{d\theta}{dr}\right]$$

(ii) Length of the normal $= PG = r\,\mathrm{cosec}\,\phi = r\sqrt{1+\cot^2\phi}$

$$= r\sqrt{1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

(iii) Length of the sub-tangent $= OT = r\tan\phi = r.r\frac{d\theta}{dr}$

$$= r^2\frac{d\theta}{dr}$$

(iv) Length of the sub-normal $= GO = r\cot\phi = r.\frac{1}{r}\frac{dr}{d\theta}$

$$= \frac{dr}{d\theta}$$

Note : If $\frac{1}{u} = r$

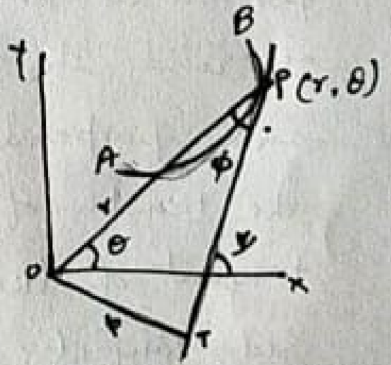
or, $-\frac{1}{u^2}\frac{du}{d\theta} = \frac{dr}{d\theta}$ (differentiating $\frac{1}{u} = r$ w.r to θ)

Then Polar sub-tangent $= -\frac{d\theta}{du}$.

Polar sub-normal $= \frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$

Perpendicular from the pole upon the tangent on the curve

Let P be any point on the curve AB whose polar coordinates be (r,θ) so that $OP = r$ and $\angle POX = \theta$, Let PT be the tangent at P on the curve. Let ϕ be the angle of the tangent which makes with the radius vector. Let OT (= P) be the length of the perpendicular from the pole O upon the tangent PT then from Δ OPT. $p = r\sin\phi$ or, $\frac{1}{p} = \frac{1}{r\sin\phi}$



or, $\frac{1}{p^2} = \frac{1}{r^2\sin^2\phi} = \frac{1}{r^2}.\mathrm{cosec}^2\phi = \frac{1}{r^2}(1+\cot^2\phi)$

$= \frac{1}{r^2}\left\{1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\}$ $\quad\therefore \frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

Remark If $u = \frac{1}{r}$ or, $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$, then

$$\frac{1}{p^2} = u^2+\left(\frac{du}{d\theta}\right)^2$$

This is another form of p.